Contents

CHAPTER 1

Equations and Inequalities

This chapter describes ways to gently introduce the idea of solving equations and finding the values of unknown quantities. The chapter starts off with two sections on verbal puzzles that you and your child can give each other. These puzzles are equations in disguise.

The next two sections begin the transition to written problems. They use blank spaces or letter substitution puzzles to accustom your child to working with unknowns. Letter substitution puzzles are used as first examples of using variables for unknowns. The idea of substituting a single digit for a single letter is easily understandable for most children, and it is less abstract and intimidating than solving for a variable in an equation.

After this progression of warm-up sections, one-step equations are introduced. These equations require one arithmetic step to solve. Your child will learn to use reverse operations to make these equations easy to solve. The use of reverse operations will be carried further for two-step equations a couple of sections later.

This is a good time to start doing word problems with your child, so you may want to start going through the material in Chapter 2: *Word Problems*. Word problems are a natural source of equations, so practicing them will show the usefulness of learning how to solve equations.

The second half of the chapter is a significant step up in sophistication from the first half. Be patient and make sure that your child has thoroughly mastered all of the skills in the first half of the chapter before moving on to this material. If your child has some significant problems with this new material, consider waiting 6 to 12 months before attempting it again. The ability to handle increased levels of abstraction and working with symbols in place of numbers is a developmental step—it cannot be rushed.

The next three sections cover skills needed to solve two-step equations. The first of these describes the order used to evaluate complex mathematical expressions. These rules are needed for deciding the order to use for solving two-step equations. There is a section on working with the distributive property, which is a property often needed when working with more complex expressions. The process of solving two-step equations is described as peeling an onion—reverse operations are used to peel back the onion one layer at a time to discover the value of the variable.

After that there are two sections describing basic ways to organize and clean up equations. The first step is to consolidate the variables on each side of an equation. The second section shows how to take variables on both sides of an equation, and move them all to one side.

The chapter then covers the most basic case of having several variables in several equations. This section introduces the idea of substitution, which means replacing a quantity by something that is equal to it. The idea is to get rid of the extra variables by using substitution.

The last section on equations covers solving proportions. Proportions occur frequently in ratio and rate problems, and also in geometry scaling problems. Proportions involve equations of a particular kind, and I describe a couple of methods for solving them.

After all of this work on equations, the last section of the chapter discusses inequalities. It starts off by describing the various kinds of inequalities, and how they can be displayed on a number line. It then talks about how inequalities can be solved using exactly the same techniques your child has learned to solve equations.

1.1 I'm thinking of a number

LESSON

Learning about beginning equations using verbal puzzles.

GAME **I'm Thinking of a Number**: One person says he is thinking of a number. That person then says a mathematical clue concerning his number. For example, "When I add 15 to my number I get 29—what is my number?"

Practice

Use reverse operations Continuing the example in the game description above, we want to find the number for, "when I add 15 to my number I get 29." To reverse the adding of 15, you should subtract 15. So the number must be $29 - 15 = 14$.

Encourage your child to check the answer by putting it back in the original problem. In this case, check the answer by verifying that $14 + 15 = 29$.

Silly names Introduce the use of variable names by making the following change to the game. Tell your child that you are getting tired of referring to your number as "my number" or "the number I am thinking of." Say that you want to make it shorter, and to name it after something. Pick any silly name you want—the name of a doll, a friend, or whatever.

Now the game becomes—"I'm thinking of a number called Pluto. When I subtract 34 from Pluto I get 22. What is Pluto?"

Switch roles, play anywhere Switch roles with your child from time to time. Having your child ask the questions to challenge you is not only fun for your child, but the forming of the question, and checking to make sure that you get the right answer, leads to a better understanding of the whole process.

This game, and the bag game introduced in the next section, are great games to play in the car, bus, or train as you take the kids on various errands and activities.

2.10 Problem solving strategies

LESSON

Learning about problem solving strategies.

Practice

Almost all beginning word problems can be solved using the five-step method, and it is a very useful tool. However, it will not automatically solve every word problem you come across. In particular, the second and third steps of the method, those that involve translating sentences into math and then solving that math, can be difficult to do for certain problems.

This section contains many of my favorite problem solving strategies. There are entire books written on this topic, so this section will only serve as an introduction to the ideas. If you are interested in looking more deeply into this topic, you might, for example, want to look at the book *Creative Problem Solving in School Mathematics* by Dr. George Lenchner.

Be persistent This is the most important of these strategies. Studies have found a strong correlation between persistence in problem solving and mathematical ability.

Most children will give up on a math problem in less than a minute. They will quickly turn to you and ask for help or even the answer. If you give them the answer, you are reinforcing that your child could not do the problem and that it was a good idea to give up.

Help your child learn persistence by aiding the exploration for the answer without giving it away. Guide them through different approaches to the problem. Show them that problem solving can be viewed as a challenge, that a problem is a fun puzzle or mystery to be unraveled and solved. Teach them to be willing to take a break from a problem and then come back to it for a second try.

Find similar problems you know how to do Ask: "Which problem have you seen (and already know how to do) that reminds you of this problem?" If your child, or you, can identify such a problem, then ask if the method for solving the old problem will work for this new problem.

Learn from simpler problems This is one of my favorite strategies, and it is very useful for non-mathematical problems. There are two ways to use this idea.

One approach is to look for simpler problems that are similar to the current problem. I use this strategy a great deal when tutoring students who are confronting a type of problem that is more complex than what they are used to.

For example, suppose you ask what the average speed is if you have traveled 30 miles in three-fourths of an hour. Your child may be put off by seeing a fraction, $\frac{3}{4}$, and feel confused. In this case, ask your child how to do a problem without a fraction—ask what the average speed is if you traveled 30 miles in 2 hours. Change as little as you can about the problem to make it into one that is familiar. If your child knows to solve this new problem by dividing 30 by 2, ask if that same method can be used to solve the earlier problem.

The second approach is to break the original problem into simpler pieces. For example, suppose you are asked to solve the counting problem at the start of Section 4.7: *Independent vs. exclusive*. This involves counting the number of ways of getting from town A to town D. If you separate this problem into counting the number of ways of going from A to B to D, and counting the number of ways of going from A to C to D, then you have broken the original problem into two problems that are much easier to do.

Do examples and look for patterns Sometimes you look at a problem and have no idea how to solve it in a general way. If you can find a way to work with smaller versions of the original problem, you may be able to find patterns in these examples that will lead to the solution of the general problem. This is an important special case of the previous strategy.

For example, what is the sum of the first 20 odd numbers? This is a problem that young children do not have the tools to solve all at once. However, it becomes easy if you are willing to do smaller examples and look for a pattern. The sum of 1 odd number is 1, of 2 odd numbers is $1 + 3 = 4$, of 3 odd numbers is $1 + 3 + 5 = 9$, of 4 odd numbers is 16, and of 5 odd numbers is 25. At this point, your child may notice the pattern—that the sum is the square of the number of odd numbers you have.

This strategy develops a willingness to go in and play around with the original problem, to try things out and see where they lead, and that is a wonderful habit to develop.

Educated guess and check Doing something called *guess and check* sounds like something that someone does when they have no idea what to do. That is not the case. When done well, this is a very powerful strategy that may quickly lead to a solution that would otherwise be hard to reach. The process of experimenting with and revising the guesses will often lead to understanding how to solve the problem. It is a perfect strategy for a young child that does not have very many sophisticated math tools to use.

For example, suppose you are told that Grace spent 462 dollars buying 20 math and history books. The math books cost 21 dollars and the history books cost 28 dollars. We could solve this problem using equations and substitution, but let's use guess and check instead.

You might make an initial guess that she bought 10 of each—that would lead to a total cost of 490 dollars, which is too high. Lower the total cost by making a new guess that involves buying more of the cheaper books. If you guess that she bought 15 math books, that leads to a total cost of 455 dollars, which is just a little too low. Making a small modification leads to the correct answer: she bought 14 math books and 6 history books.

Pictures and diagrams These tools give you ways to organize and display the data from a word problem in a form that greatly improves understanding.

It is almost always a good idea to make a drawing to go with a geometry problem. If you have a problem that involves a tree and the length of its shadow, draw the situation and fill in the information you are given.

Diagrams are also very useful ways to organize data. For example, Venn diagrams are introduced and used to solve a problem in Section 3.2: *Sets and Venn diagrams*.

Charts For some problems, you may be able to create a chart that takes advantage of a general relationship present in the problem. Rate problems often have this property.

Charts are good organizational tools that make it clear which things are known, which things need to be found out, and what the relationships are. Use a chart to solve the following work rate problem. Roy takes 3 hours to mow a large lawn. Bob takes 2 hours to mow the same lawn. If Roy and Bob work together, how long will it take them to mow the lawn?

The important relationship is that rate $\times$ time = work. Here is a table that captures this relationship and the given information:

	rate x	time =	work
Roy		3	1
Bob		2	1
Together			

When faced with this chart, it is compelling to start filling in the rates: $\frac{1}{3}$ for Roy and $\frac{1}{2}$ for Bob. After that, the rate for the two working together will be the sum of their rates, so the rate when they work together is $\frac{5}{6}$. Finally, this produces a time of $\frac{6}{5}$, or 1 hour and 12 minutes, for the time it will take them to mow the lawn together.

Make a table of information This is similar to making a chart. If there is a lot of data in a problem, have your child record it in an organized way.

For example, I mentioned earlier in this section doing examples for the sum of the first few odd numbers. In doing that problem, you could have your child make a table with two columns—the first column has the number of odd numbers and the second has their sum.

It is a lot easier to see patterns in an organized chart than in notes scattered around a piece of paper.

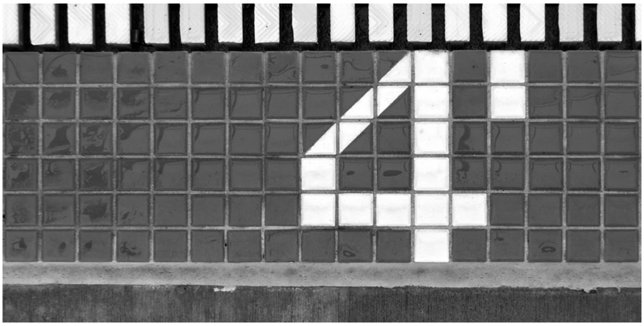
Act out the scenario For some children, it can help tremendously if they act out, either in reality or in their heads, the scenario of the problem. If there are a sequence of steps, say buying and selling something, have your child act out the steps to see how the problem will work out.

Work with others There are two things that are important about working with others. The first is that it teaches your child the value of sharing ideas—that two children may each have something valuable to offer the other. It is important for your child to learn that doing mathematics need not be an isolating pursuit. The second is that it takes practice to develop the social skills for being able to effectively work with others.

Work backwards Some problems are easier to work backwards than they are forwards.

For example, take the simple problem of finding the number that when you double it and add 5 it equals 23. You could do this by using guess and check and working forwards. However, it is easier to start at the end, 23, and work backwards—take 5 away from 23 to get 18, and then take half of 18 to get 9.

Partial solutions are valuable Most people who fail to solve a problem feel that, since they didn't solve the problem, their work on the problem must be worthless. Encourage your child to value partial solutions. Do this by starting with what has been done so far and demonstrating how it can be extended to a solution of the problem. Eventually, over many years, your child will learn to come back to partial solutions and push through the roadblocks to finish the problem—this is persistence at its best.



CHAPTER 4 Probability and Counting

This chapter contains two closely connected topics—probability and counting. These topics are interwoven throughout the chapter.

Games and chance As your child starts playing games of chance, and starts wondering how likely things are to happen, your child will start being involved with probability. Since this exploration will often be connected with games, this is usually a lot of fun.

Introduce your child to probability in simple day-to-day things. If a shirt needs to be taken out of a drawer, ask your child to try it with eyes closed, and talk about the probability of getting the right shirt.

The central theme for calculating probabilities is that the probability of something occurring is the number of ways it can occur successfully divided by the number of ways that anything can occur. For example, if there are 2 dark pairs of socks and 3 light pairs, the probability is $\frac{2}{5}$ of picking a dark pair of socks without looking. Emphasize that this formula is only true when all of the possibilities you are counting are equally likely to happen.

Likely, not guaranteed Impress upon your child that probability is not a guarantee, it is a likelihood. The probability of getting heads is $\frac{1}{2}$, but that does not mean that when you flip a coin 100 times it will be heads 50 times.

The only time an outcome is guaranteed is when the probability is 0 or 1. If the probability is 0, then the outcome cannot occur, and if the probability is 1, the outcome must occur.

Independent events can be counterintuitive Events are said to be *independent* if they do not influence each other in any way. It is difficult for some people to believe that independent events are truly independent.

If you have flipped a coin 5 times and you have seen 5 tails, many adults will say one of two things. Some will say that since heads should occur half of the time, you are overdue to get heads on the next flip. Other adults will say that there is clearly a run of tails going on, so the next flip is much more likely to be tails. Teach your child that the probability is still $\frac{1}{2}$ that it will be heads on the next flip. Independent events, such as coin flips, have no memory—the probability remains the same, no matter what the history is.

With all of the formulas and calculations, it is easy to lose sight of the feeling of the subject. Sometimes, just talk about whether something is more likely or less likely to happen. For example, look at some incoming large, dark clouds and talk about whether it seems likely to rain.

Learn the methods Teach your child the techniques and ideas behind the counting formulas. The formulas do not need to be memorized if the methods behind them are understood.

As discussed in the introduction to this book, your child will benefit from being taught many of these topics through problem solving. Rather than starting a new topic with a rule for how things are done, start with a number of example problems and see if your child can be guided to discovering the rule.

There is a style of thinking for this kind of counting that, once learned, feels natural and is quite powerful. Children enjoy the way the counting ideas fit together, and they enjoy the large numbers that are produced by a little multiplying and a few factorials.

Counting is very important to probability. The counting material in this chapter supports calculations of probability, but it is also interesting in its own right. For years my children have enjoyed being able to calculate the number of glass clinks when a group of people are at my house making a toast.

Some history Risk taking, and its associated thoughts of probability, have been around for thousands of years. The Babylonians had forms of insurance for merchant sea voyages, the Romans had annuities in which a one-time lump-sum payment could be made in exchange for a large number of periodic payments, and of course gambling has been around for all of recorded history. However, all of this risk taking was done by feeling, without any systematic way of calculating the related probabilities.

There are two independent efforts given credit for establishing a more rigorous view of probability. Gerolamo Cardano was an Italian physician, mathematician, and gambler living in the 1500's. He wrote a book on some basic principles of probability in 1525, but it was not published and recognized until 1663. Later, in 1654, there was a gambling question that led to a correspondence between two famous French mathematicians, Blaise Pascal and Pierre de Fermat. From their letters and ideas sprang the foundations for the modern view of probability.

Your child may enjoy investigating and learning about the development of probability since the time of Pascal and Fermat. Enter "probability" and "history" into any internet search engine, and you will find a wealth of sources.

Shameless plug The book *Introduction to Counting & Probability* by David Patrick, published by the Art of Problem Solving, covers the topics of this chapter in much more depth, provides a great many exercise problems, and covers many topics that are natural extensions. The interested child will find this book an excellent way to enjoy more material on this subject.

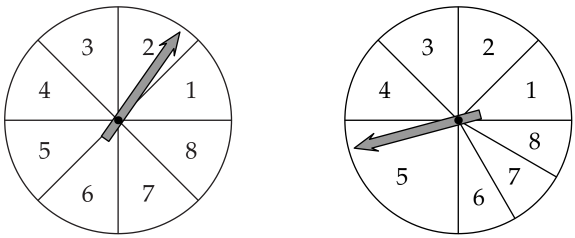
4.1 Basic probability

LESSON Learning the basic ideas of probability.

Practice **Events** An *event* is a collection of possible outcomes (possibilities).

For example, rolling an odd number with a die is an event. It consists of the three possibilities of rolling a 1, 3, or 5.

Equally likely possibilities The simplest, and most common, form of probability involves possibilities that are equally likely.



Show your child these pictures of two spinners. With the first spinner, the 8 regions are equal in size, so the spinner is equally likely to land on each one. In the second spinner, the region for the 5 is quite large, so it is far more likely to be chosen.

Definition of probability This next definition of probability only applies when equally likely possibilities are involved. I will use the notation $P(Event)$, or more briefly $P(E)$, to refer to this probability.

MATH WORDS The *probability* of an event, $P(E)$, equals

$$\frac{\text{Number of possibilities in the event}}{\text{Total number of possibilities}}$$

Counting is central to this definition of probability. Because of this, most of this chapter will be spent looking at ways to count things.

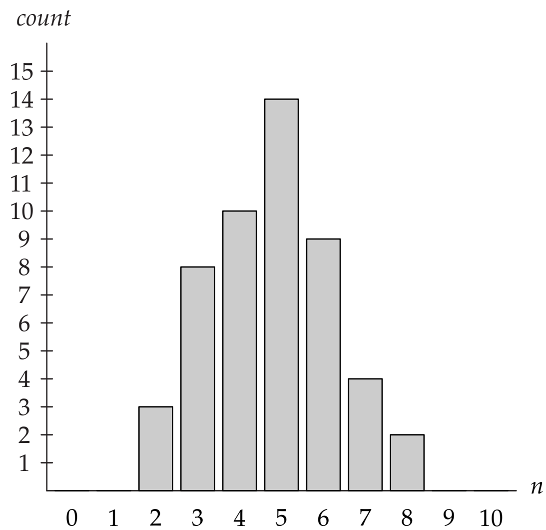
Examples When you flip a coin there are two possibilities that are equally likely—heads and tails. There is only one way to get a head, so the probability of getting a head is $\frac{1}{2}$.

As another example, if you roll one die, what is the probability that you will get less than a 3? With a fair die, the six numbers are equally likely. There are two ways to be successful—to roll a 1 or a 2. There are a total of six possibilities. So, the probability is $\frac{2}{6}$, which is $\frac{1}{3}$.

Range of probability values The value of any probability is somewhere from 0 to 1.

If the probability is 0, that means the event is impossible. The probability of rolling a 7 with one die is 0. If the probability is 1, that means that the event is certain to happen. The probability is 1 of rolling a number that is somewhere from 1 to 6.

No guarantees Unless the probability of an event is 0 or 1, there is no guarantee that the event will occur as often as the probability indicates. For example, the probability of getting heads with a flipped fair coin is $\frac{1}{2}$. However, if you flip a coin 2 times, you will often see a total of 0 heads or 2 heads.



This histogram shows the number of times n heads occurred when I flipped a group of 10 pennies 50 times.

Make graphs of your own experiments Do experiments like this last one with your child. Flip a coin 10 times and record how many heads you see. Do this as many times as your child is interested. You may want to do a frequency graph, covered in Section 5.2: *Tables of data*, to make the results easier to look at. You will probably see that getting 4, 5, and 6 heads is most common, but that other numbers happen as well.

A similar experiment is to keep track of dice rolls. Make a bar graph on some graph paper for the numbers from 2 to 12. Each time your child rolls the dice, darken in one more square in the column above the number for the sum that was rolled. At first the height of the columns will not form much of a pattern. With enough rolls you should see a flowing curve with 7 occurring the most in the middle. Discuss with your child why it took a large number of rolls before it started to look like a nice curve.

Practice while playing games Have your child practice these ideas in any game that involves picking a number at random—whether it uses cards, spinners, or dice. Before some of the turns, have your child calculate the probability of getting the desired numbers, or the probability of avoiding the bad numbers.

RELATED STEPS

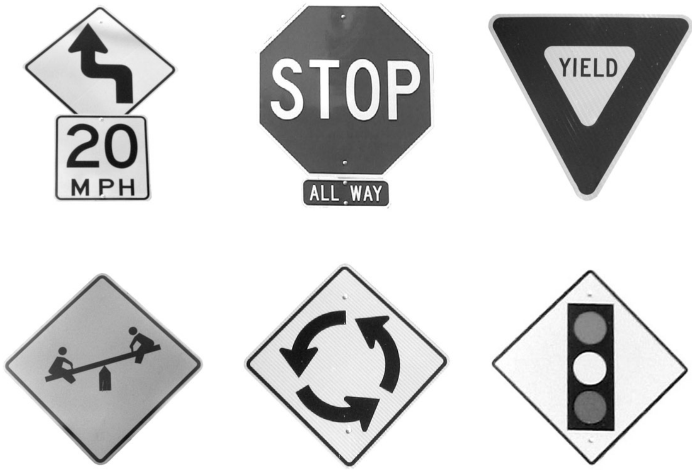
There are many times when your child will be gathering and displaying data, and those are excellent opportunities to practice the skills in Chapter 5: *Statistics and Graphing*.

CHAPTER 6 Geometry

Beginning geometry is all about drawing, creating, and playing with all sorts of shapes. In addition to this interactive and playful sense of objects, part of the mathematics to be learned is the names and descriptions of shapes. This builds an awareness in your child of these objects, and it creates a vocabulary that your child can use to think about and discuss geometry in everyday things.

Traditional playing with blocks is a natural and good source of experience with geometry. Seeing how blocks fit together, and don't fit together, is one of the best ways for your child to learn the characteristics of various shapes and sizes. The famous architect Frank Lloyd Wright as a child played with a special set of blocks designed by Friedrich Froebel, and he credits this experience with giving him insights into geometry, mathematics, and architecture.

Shapes Start at a young age by teaching your child the names for basic shapes such as triangles, rectangles, squares, and circles. This teaching is not done in any formal way: just use the names when referring to everyday things and your child will learn them.



As your child is ready, transition to some of the names for more complex shapes, and start using some of the concepts introduced later in this chapter. Make sure your child's toys include a wide variety of shapes to play with, including shapes with more than four sides.

Vocabulary games There is a lot of terminology needed to be able to talk about geometric ideas, and it can seem somewhat endless at times. There are some ways to make a game out of learning all of this vocabulary.

One way is to have a scavenger hunt. Start by making a list of geometric ideas you are searching for. This can be names of two-dimensional or three-dimensional shapes, or it can be features such as having symmetries or right angles. The first person to find all of the items on the list wins.

Another game to play is Concentration. Make a set of cards that have vocabulary words and drawings of shapes. The cards are placed face down, and players take turns turning over a pair of cards looking for a match. If there is a match, the cards are kept and the player continues. If the cards do not match, the cards are turned back face down, and the next player goes.

Concentration can be played in many ways. One way is to have two cards for each idea, and a match is an exact match—word with word, drawing with drawing, or word with drawing. Another way is to have just one copy of most cards, and matches are made by being able to name one (or possibly two) features the two cards have in common. For example, a square and a rectangle might match because they both have right angles and are quadrilaterals.

Yet another way to play with this is to use a collection of shapes. Use blocks already made into interesting shapes, or use pieces of stiff cardboard you have cut into various shapes. With eyes closed, have your child choose a piece from a container and describe it using the new vocabulary.

Patterns, symmetries, and transformations One way for your child to see the world through geometric eyes is to look for symmetries and the effects of transformations (these are described in sections 6.9 and 6.10). Thinking about geometry from this point of view teaches your child to see things not just as they are, but to explore what can happen to them as they are mentally moved around and manipulated.

Several of the sections of this chapter describe patterns and designs that your child can play with and explore. You can look at or construct tiling patterns and drawing designs. You can find pictures of patterns in weaving, pottery, and architecture. Many of these designs and patterns are rich playgrounds for exploring how shapes interact, and most have interesting symmetries.

Activities This chapter is filled with terminology and lists of properties. You should not view this as a large collection of facts to be memorized and recited. Try to maintain the view that these are related concepts that can be discovered, dipped into, and played with.

A good way to bring these ideas alive is to engage in related activities. Section 6.4: *Taking directions* describes a game for practicing basic distances and directions. For those willing to dip into some programming, there is the LOGO environment in which beautiful pictures can be created using elementary geometry ideas.

Another activity is making straight-edge and compass constructions. These can be fun, but they are somewhat limited in number. You can extend these constructions by using a straight-edge and compass to create geometric designs for drawings. A wonderful, dynamic way to play with these ideas is to use either of two software programs designed for this purpose: *GeoGebra* or *Geometer's Sketchpad*. These ideas are discussed further in Section 6.16: *Straight-edge and compass*.

Graph theory The graph theory section at the end of the chapter creates a very different geometric world for your child to play in. There are a number of graph theory related puzzles that your child can play with.

Shameless plug The book *Introduction to Geometry*, which is written by Richard Rusczyk and published by the Art of Problem Solving, covers the topics of this chapter in much more depth, and it covers many topics that are natural extensions. The book can even serve as a textbook for a full geometry course. The interested child will find it an excellent way to enjoy more material on this subject.